

# The Universal $R$ -Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial.

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## Abstract

P. Melvin and H. Morton [9] studied the expansion of the colored Jones polynomial of a knot in powers of  $\check{q} - 1$  and color. They conjectured an upper bound on the power of color versus the power of  $\check{q} - 1$ . They also conjectured that the bounding line in their expansion generated the inverse Alexander-Conway polynomial. These conjectures were proved by D. Bar-Natan and S. Garoufalidis [1].

We have conjectured [12] that other ‘lines’ in the Melvin-Morton expansion are generated by rational functions with integer coefficients whose denominators are powers of the Alexander-Conway polynomial. Here we prove this conjecture by using the  $R$ -matrix formula for the colored Jones polynomial and presenting the universal  $R$ -matrix as a ‘perturbed’ Burau matrix.

# 1 Introduction

Let  $\mathcal{K}$  be a knot in  $S^3$  endowed with canonical framing (*i.e.*, its self-linking number is zero). We assign to this knot an  $\alpha$ -dimensional  $SU_q(2)$  module  $V_\alpha$ .  $J_\alpha(\mathcal{K}; \check{q})$  denotes the colored Jones polynomial of  $\mathcal{K}$ , normalized in such a way that it is multiplicative under a disconnected sum and

$$J_\alpha(\text{unknot}; \check{q}) = \frac{\check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}}}{\check{q}^{\frac{1}{2}} - \check{q}^{-\frac{1}{2}}}. \quad (1.1)$$

Another popular normalization for the Jones polynomial is

$$V_\alpha(\mathcal{K}; \check{q}) = \frac{J_\alpha(\mathcal{K}; \check{q})}{J_\alpha(\text{unknot}; \check{q})} = \frac{\check{q}^{\frac{1}{2}} - \check{q}^{-\frac{1}{2}}}{\check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}}} J_\alpha(\mathcal{K}; \check{q}). \quad (1.2)$$

The advantage of the normalization (1.2) is that

$$V_\alpha(\mathcal{K}; \check{q}) \in \mathbb{Z}[\check{q}, \check{q}^{-1}]. \quad (1.3)$$

The colored Jones polynomial is an effective invariant of knots. However, its relation to classical topological invariants of knots remains mostly obscure (see, *e.g.* a review [2]). One may try to decompose  $V_\alpha(\mathcal{K}; \check{q})$  into some simpler ‘building blocks’ in a hope that their topological nature would be easier to establish. An important step in this direction was made by P. Melvin and H. Morton [9]. They suggested to expand the Jones polynomial in powers of  $\alpha$  and

$$h = \check{q} - 1 \quad (1.4)$$

(actually, they expanded  $J_\alpha(\mathcal{K}; \check{q})$  and used  $\log(1 + h)$  rather than  $h$  as an expansion parameter; this seemed to be a more ‘physically natural’ choice). If we expand  $V_\alpha(\mathcal{K}; \check{q})$  in Taylor series in  $h$  for a fixed value of  $\alpha$  then, according to [9], the coefficients will be finite degree polynomials in  $\alpha$ :

$$V_\alpha(\mathcal{K}; \check{q}) = \sum_{n \geq 0} h^n \left( \sum_{0 \leq m \leq n} D_{m,n}(\mathcal{K}) \alpha^{2m} \right). \quad (1.5)$$

Melvin and Morton proved that the coefficients  $D_{m,n}(\mathcal{K})$  are finite type invariants of  $\mathcal{K}$  of order  $n$ . Moreover, they conjectured that the bound on powers of polynomials of  $\alpha$  can

be improved and that the bounding line contribution to  $V_\alpha(\mathcal{K}; \check{q})$  is equal to the inverse Alexander-Conway polynomial:

**Theorem 1.1** *For a knot  $\mathcal{K} \subset S^3$  the coefficients  $D_{m,n}(\mathcal{K})$  of the expansion (1.5) satisfy the following two properties:*

$$D_{m,n}(\mathcal{K}) = 0 \quad \text{for } m > \frac{n}{2}, \quad (1.6)$$

$$\sum_{m \geq 0} D_{m,2m}(\mathcal{K}) a^{2m} = \frac{1}{\Delta_A(\mathcal{K}; e^{i\pi a} - e^{-i\pi a})}, \quad (1.7)$$

here  $a$  is a formal parameter and  $\Delta_A(\mathcal{K}; z)$  is the Alexander-Conway polynomial of  $\mathcal{K}$  which satisfies the skein relation of Fig. 1 and is normalized by the condition

$$\Delta_A(\text{unknot}; z) = 1. \quad (1.8)$$

This theorem was proved by D. Bar-Natan and S. Garoufalidis [1].

$$\Delta_A \left( \begin{array}{c} \diagup \diagdown \\ \text{circle} \end{array}; z \right) - \Delta_A \left( \begin{array}{c} \diagup \diagup \\ \text{circle} \end{array}; z \right) = z \Delta_A \left( \begin{array}{c} \text{circle} \\ \text{circle} \end{array}; z \right).$$

Figure 1: The skein relation for the Alexander-Conway polynomial

In [12] we conjectured that the expansion (1.5) satisfies some further properties. To formulate our conjecture we rearrange eq. (1.5) as a formal power series in  $h$  and  $\alpha h$ :

$$V_\alpha(\mathcal{K}; \check{q}) = \sum_{n \geq 0} h^n \sum_{m \geq 0} D_{m,n+2m} (\alpha h)^{2m}. \quad (1.9)$$

Then, motivated by eq. (1.7), we introduce a new variable

$$z = \check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}} \quad (1.10)$$

instead of  $\alpha h$ :

$$\alpha h = 2 \log \left( \sqrt{1 + \left( \frac{z}{2} \right)^2} + \frac{z}{2} \right) \frac{h}{\log(1+h)} = z + \mathcal{O}(z^3, h). \quad (1.11)$$

Substituting eq. (1.11) into eq. (1.9) as a formal power series we get the expansion in powers of  $h$  and  $z^2$ :

$$V_\alpha(\mathcal{K}; \tilde{q}) = \sum_{n \geq 0} V^{(n)}(\mathcal{K}; z) h^n, \quad (1.12)$$

$$V^{(n)}(\mathcal{K}; z) = \sum_{m \geq 0} d_m^{(n)}(\mathcal{K}) z^{2m}. \quad (1.13)$$

Eq. (1.7) implies that the formal power series  $V^{(0)}(\mathcal{K}; z)$  comes from the Taylor expansion of the inverse Alexander-Conway polynomial at  $z = 0$ :

$$V^{(0)}(\mathcal{K}; z) = \frac{1}{\Delta_A(\mathcal{K}; z)}. \quad (1.14)$$

We conjectured ('strong conjecture' of [12]) that the 'upper lines'  $V^{(n)}(\mathcal{K}; z)$  of the expansion (1.12) come also from the Taylor expansion of rational functions of  $z$ .

**Theorem 1.2** *There exists a set of polynomial invariants of knots in  $S^3$*

$$P^{(n)}(\mathcal{K}; z) \in \mathbb{Z}[z^2], \quad (1.15)$$

such that the series  $V^{(n)}(\mathcal{K}; z)$  is the Taylor series expansion at  $z = 0$  of the rational function

$$V^{(n)}(\mathcal{K}; z) = \frac{P^{(n)}(\mathcal{K}; z)}{\Delta_A^{2n+1}(\mathcal{K}; z)}. \quad (1.16)$$

In [12] we presented experimental evidence in support of this theorem by calculating the first polynomials  $P^{(n)}(\mathcal{K}; z)$  of some simple knots.

The Alexander-Conway polynomial of a knot satisfies the properties

$$\Delta_A(\mathcal{K}; z) \in \mathbb{Z}[z^2], \quad (1.17)$$

$$\Delta_A(\mathcal{K}; 0) = 1. \quad (1.18)$$

Therefore as a Taylor series expansion at  $z = 0$ ,

$$\frac{1}{\Delta_A(\mathcal{K}; z)} \in \mathbb{Z}[[z^2]], \quad (1.19)$$

and in view of eq. (1.15) we have the following simple corollary of the Theorem 1.2 ('weak conjecture' of [12]):

**Corollary 1.1** All the coefficients  $d_m^{(n)}(\mathcal{K})$  of the expansion (1.13) are integer:

$$d_m^{(n)}(\mathcal{K}) \in \mathbb{Z}. \quad (1.20)$$

Note that this corollary is stronger than the condition

$$n! D_{m,n}(\mathcal{K}) \in \mathbb{Z}, \quad (1.21)$$

which comes from the fact that because of (1.3)

$$\sum_{\substack{0 \leq m \leq n \\ 0 \leq m \leq \frac{n}{2}}} D_{m,n}(\mathcal{K}) \alpha^{2m} \in \mathbb{Z} \quad \text{for } \alpha \in \mathbb{Z}. \quad (1.22)$$

R. Lawrence [8] has formulated a conjecture about  $p$ -adic properties of Ohtsuki's invariants of rational homology spheres. We proved that conjecture in [13] for the special case of a manifold constructed by a rational surgery on a knot in  $S^3$ . Our proof was based on Corollary 1.1 which we used as a conjecture. Thus the proof of Theorem 1.1 that will be presented in this paper, completes the proof of [13].

We will prove Theorem 1.2 in the following equivalent form:

**Proposition 1.1** For a knot  $\mathcal{K} \subset S^3$  there exists a set of polynomials  $P^{(n)}(\mathcal{K}; z) \in \mathbb{Z}[z^2]$  such that for a fixed  $\alpha$  and for any  $M > 0$

$$\sum_{\substack{0 \leq n \leq M \\ 0 \leq m \leq \frac{n}{2}}} D_{m,n}(\mathcal{K}) \alpha^{2m} h^n = \sum_{0 \leq n \leq M} h^n \frac{P^{(n)}(\mathcal{K}; (1+h)^{\frac{\alpha}{2}} - (1+h)^{-\frac{\alpha}{2}})}{\Delta_A^{2n+1}(\mathcal{K}; (1+h)^{\frac{\alpha}{2}} - (1+h)^{-\frac{\alpha}{2}})} + \mathcal{O}(h^{M+1}). \quad (1.23)$$

The equivalence between this proposition and Theorem 1.2 follows from the structure of the substitution (1.11). A coefficient  $d_m^{(n)}$  of the expansion (1.13) is determined only by the coefficients  $D_{m',n'}$ ,  $m' \leq m$ ,  $n' \leq n + 2m$ . Therefore eq. (1.23) for  $M = n + 2m$  indicates that the coefficient  $d_m^{(n)}$  does come from the Taylor expansion of the r.h.s. of eq. (1.16) at  $z = 0$ .

*Outline of the Proof of Proposition 1.1.* Similarly to [9], we use the expression for the colored Jones polynomial  $J_\alpha(\mathcal{K}; \check{q})$  as a (quantum) trace of the product of  $\check{R}$ -matrices. The matrices

correspond to elementary braids in the picture of the knot  $\mathcal{K}$  as a closure of an  $N$ -strand braid  $B_N$ . The trace is taken over the tensor product  $V_\alpha^{\otimes N}$  of the  $\alpha$ -dimensional  $SU_q(2)$  modules  $V_\alpha$ . A choice of a particular basis  $f_m$ ,  $0 \leq m \leq \alpha - 1$  in  $V_\alpha$  allows us to map  $V_\alpha^{\otimes N}$  into the algebra  $\mathbb{C}[z_1, \dots, z_N]$  of polynomials of  $N$  variables:

$$f_{m_1} \otimes \cdots \otimes f_{m_N} \rightarrow z_1^{m_1} \cdots z_N^{m_N}. \quad (1.24)$$

The action of  $\check{R}$ -matrices on  $V_\alpha^{\otimes N}$  can be extended to the space  $\mathbb{C}[z_1, \dots, z_N]$ . It turns out that in the approximation, when  $h \rightarrow 0$  and  $\check{q}^\alpha$  is kept constant, the action of the  $\check{R}$ -matrix becomes an endomorphism of the algebra  $\mathbb{C}[z_1, \dots, z_N]$ . This endomorphism is generated by a linear transformation of the space  $\mathbb{C}^N$  of variables  $z_1, \dots, z_N$ . This transformation coincides with the Burau representation of elementary braids.

Let  $\mathcal{O}$  be such an endomorphism of  $\mathbb{C}[z_1, \dots, z_N]$ . Its trace over the whole algebra  $\mathbb{C}[z_1, \dots, z_N]$  can be expressed in terms of the restriction  $\tilde{\mathcal{O}}$  of the action of  $\mathcal{O}$  to the subspace  $\mathbb{C}^N$  of variables  $z_1, \dots, z_N$ . More precisely,

$$\text{Tr}_{\mathbb{C}[z_1, \dots, z_N]} \mathcal{O} = \frac{1}{\det_{\mathbb{C}^N}(1 - \tilde{\mathcal{O}})}. \quad (1.25)$$

Since the matrix  $\mathcal{O}$  coming from the braid  $B_N$  coincides with the Burau representation, the determinant in the denominator of the *r.h.s.* of eq. (1.25) is equal to the Alexander-Conway polynomial of  $\mathcal{K}$ . This leads to the Melvin-Morton relation (1.14). Since we never use the bound (1.6), these considerations constitute yet another,  $R$ -matrix based, proof of the Melvin-Morton conjecture.

The exact  $\check{R}$ -matrix presented as a series in  $h$  is not an endomorphism of  $\mathbb{C}[z_1, \dots, z_N]$ . The coefficients at higher powers of  $h$  are rather ‘perturbed endomorphisms’ whose traces can be calculated with the help of standard tricks of Quantum Field Theory. These tricks are rigorous in our finite-dimensional context. The result of such calculation is the expansion (1.12), (1.13), (1.16).

The fact that Burau representation generates the  $R$ -matrix based braid group representation in the limit of  $h \rightarrow 0$ ,  $\check{q}^\alpha = \text{const}$ , can be traced back to the equivalence between

the  $R$ -matrix representation and the action of the braid group on the twisted  $m$ th cohomology of the configuration space of  $m$  points on a complex plane with  $N$  holes, considered by R. Lawrence [6], [7] (see also the book [14], this construction is known in physical literature as ‘free field representation’).

Although we try deliberately to avoid any ‘physical’ references in this paper, we can not help but mention that our calculations are very similar to those of [4]. L. Kauffman and H. Saleur related the Burau representation to the evolution of fermionic particles along the strands of the braids. The particles scatter at the elementary braids. The scattering is described by the Burau matrix. It is essential that the particles are free, that is, they scatter independently of one another. It follows from our calculations that in the limit of  $h \rightarrow 0$ ,  $\check{q}^\alpha$ -fixed, the colored Jones polynomial comes from the evolution of ‘almost free’ bosonic particles. The ‘particles’ are monomials  $z_j$ ,  $1 \leq j \leq N$  of  $\mathbb{C}[z_1, \dots, z_N]$ , and their ‘freedom’ is the physical equivalent of the mathematical notion of endomorphism. The leading term (1.14) corresponds to absolutely free particles and expresses the fact that fermions are ‘inverse’ bosons. The subsequent terms (1.16) come from the perturbative expansion in the weak coupling constant  $h$ .

Here is the plan of the paper. In Section 2 we recall the formula for the colored Jones polynomial as a trace of the product of  $\check{R}$ -matrices over the tensor product  $V_\alpha^{\otimes N}$  of  $SU_q(2)$  modules  $V_\alpha$ . We break a strand in the braid closure and also extend the trace to the product of Verma modules  $V_{\alpha,\infty}^{\otimes N}$ . In Section 3 we discuss the expansion of the  $\check{R}$ -matrix in the limit of  $h \rightarrow 0$ . We express the terms of this expansion as derivatives of the ‘parametrized’  $\check{R}$ -matrix. In Section 4 we present parametrized  $\check{R}$ -matrix as an endomorphism of the algebra  $\mathbb{C}[z_1, \dots, z_N]$  and calculate its trace. In Section 5 we combine the results of all the previous sections and complete the proof of the Proposition 1.1. Discussion contains some comments about generalizing our calculations to links. In Appendix we prove some technical lemmas which were needed in Section 3.

## 2 The colored Jones polynomial as a character of the braid group

In this section we review the representation of the braid group based on the universal  $R$ -matrix ([10], see also a nice exposition in [5]). We also recall a formula for the colored Jones polynomial as a trace of this braid representation with one broken closure strand.

### 2.1 The $SU_q(2)$ $R$ -matrix

Let  $V_\alpha$  be the  $\alpha$ -dimensional  $SU_q(2)$  module. We choose the basis vectors  $f_m$ ,  $0 \leq m \leq \alpha - 1$  of  $V_\alpha$  in such a way that the action of the standard  $SU_q(2)$  generators is

$$X f_m = [m] f_{m-1}, \quad (2.1)$$

$$Y f_m = [\alpha - 1 - m] f_{m+1}, \quad (2.2)$$

$$H f_m = (\alpha - 1 - 2m) f_m, \quad (2.3)$$

here we use the notation

$$[n] = \frac{\check{q}^{\frac{n}{2}} - \check{q}^{-\frac{n}{2}}}{\check{q}^{\frac{1}{2}} - \check{q}^{-\frac{1}{2}}}. \quad (2.4)$$

Our basis vectors  $f_m$  are related to the basis vectors  $e_j$  of [5], eq.(2.8):

$$f_m = \frac{[\alpha - m - 1]! [m]!}{[\alpha - 1]!} e_{\frac{\alpha-1}{2}-m}, \quad (2.5)$$

here

$$[n]! = \begin{cases} \prod_{1 \leq l \leq n} [l] & \text{if } n \geq 1 \\ 1 & \text{if } n = 0. \end{cases} \quad (2.6)$$

The action of the  $R$ -matrix on the basis vectors  $f_{m_1} \otimes f_{m_2}$  of the tensor product  $V_\alpha \otimes V_\alpha$  is given by the formula which comes from substituting eq. (2.5) into Corollary 2.3.2 of [5]:

$$\begin{aligned} R(f_{m_1} \otimes f_{m_2}) &= \sum_{0 \leq n \leq m_1} \left( \check{q}^{\frac{1}{2}} - \check{q}^{-\frac{1}{2}} \right)^n \frac{[\alpha - m_2 - 1]!}{[\alpha - m_2 - n - 1]!} \frac{[m_1]!}{[m_1 - n]! [n]!} \\ &\quad \times \check{q}^{\frac{1}{4}[\alpha - 2m_1 - 1](\alpha - 2m_2 - 1) + 2n(m_1 - m_2) - n(n+1)]} f_{m_1-n} \otimes f_{m_2+n}. \end{aligned} \quad (2.7)$$

After some simple transformations this formula becomes

$$R(f_{m_1} \otimes f_{m_2}) = \check{q}^{\frac{1}{4}(\alpha^2-1)} \check{q}^{-\frac{1}{2}(\alpha-1)} \sum_{n \geq 0} \prod_{m_2+1 \leq l \leq m_2+n} (\check{q}^{-l} - \check{q}^{-\alpha}) \frac{\prod_{m_1-n+1 \leq l \leq m_1} (\check{q}^l - 1)}{\prod_{1 \leq l \leq n} (\check{q}^l - 1)} \\ \times \check{q}^{-\frac{1}{2}(\alpha-1)(m_1+m_2-n)} f_{m_1-n} \otimes f_{m_2+n}. \quad (2.8)$$

Note that in our conventions  $\sum_{a \leq j \leq b} (\dots) = 0$  and  $\prod_{a \leq j \leq b} (\dots) = 1$  if  $a > b$ .

The ‘flipped’ matrix  $\check{R}$  is defined as

$$\check{R} = PR, \quad (2.9)$$

here  $P$  is the permutation operator:

$$P(f_{m_1} \otimes f_{m_2}) = f_{m_2} \otimes f_{m_1}. \quad (2.10)$$

In order to simplify the future calculations we will write the matrix elements of  $\check{R}$  in the basis with ‘rotated phases’:

$$f_{m_1} \otimes f_{m_2} \rightarrow (\check{q}^{\frac{1}{2}(\alpha-1)})^{m_2} f_{m_1} \otimes f_{m_2}. \quad (2.11)$$

In other words, our  $\check{R}$ -matrix is defined as

$$\check{R} = (\check{q}^{-\frac{\alpha-1}{4}})^{I \otimes H} PR (\check{q}^{\frac{\alpha-1}{4}})^{I \otimes H} \quad (2.12)$$

( $I$  is the identity operator) instead of eq. (2.8):

$$\check{R}(f_{m_1} \otimes f_{m_2}) = \check{q}^{\frac{1}{4}(\alpha^2-1)} \check{q}^{-\frac{1}{2}(\alpha-1)} \sum_{n \geq 0} \prod_{m_2+1 \leq l \leq m_2+n} (\check{q}^{-l} - \check{q}^{-\alpha}) \frac{\prod_{m_1-n+1 \leq l \leq m_1} (\check{q}^l - 1)}{\prod_{1 \leq l \leq n} (\check{q}^l - 1)} \\ \times \check{q}^{-(\alpha-1)m_2} f_{m_2+n} \otimes f_{m_1-n}. \quad (2.13)$$

The elements of the inverse  $\check{R}$ -matrix can be obtained with the help of relation

$$\check{R}^{-1}(\check{q}) = P \check{R}(\check{q}^{-1}) P. \quad (2.14)$$

In other words, we have to substitute

$$f_{m_1} \otimes f_{m_2} \rightarrow f_{m_2} \otimes f_{m_1}, \quad f_{m_2+n} \otimes f_{m_1-n} \rightarrow f_{m_1-n} \otimes f_{m_2+n} \quad (2.15)$$

and  $\check{q} \rightarrow \check{q}^{-1}$  in eq. (2.13). As a result,

$$\check{R}^{-1}(f_{m_1} \otimes f_{m_2}) = \check{q}^{-\frac{1}{4}(\alpha^2-1)} \check{q}^{\frac{1}{2}(\alpha-1)} \sum_{n \geq 0} \prod_{m_1+1 \leq l \leq m_1+n} (\check{q}^l - \check{q}^\alpha) \frac{\prod_{m_2-n+1 \leq l \leq m_2} (\check{q}^{-l} - 1)}{\prod_{1 \leq l \leq n} (\check{q}^{-l} - 1)} \\ \times \check{q}^{(\alpha-1)m_1} f_{m_2-n} \otimes f_{m_1+n}. \quad (2.16)$$

## 2.2 Braid group representation and its character

Let  $B_N$  be a braid of  $N$  strands. We associate with it a tensor product  $V_\alpha^{\otimes N}$  of  $N$   $SU_q(2)$  modules  $V_\alpha$ , one module per each position. To an elementary positive braid  $\sigma_{j,j+1}$  which switches the strands at  $j$ th and  $(j+1)$ st positions we associate the  $\check{R}$ -matrix acting on  $V_\alpha \otimes V_\alpha$  at  $j$ th and  $(j+1)$ st positions in  $V_\alpha^{\otimes N}$ . The operator  $\check{R}^{-1}$  is associated to a negative elementary braid  $\sigma^{-1}$ . This construction is known to represent the braid group in  $V_\alpha^{\otimes N}$  in such a way that the action of braids commutes with the ‘global’ (*i.e.*, acting simultaneously on all individual spaces  $V_\alpha$  in  $V_\alpha^{\otimes N}$ ) action of  $SU_q(2)$ . We denote the representation of a braid  $B_N$  in  $V_\alpha^{\otimes N}$  as  $\hat{B}_N$ . Note that because of eq. (2.12), our representation is a conjugation of the standard one by the operator

$$\left(\check{q}^{\frac{\alpha-1}{4}}\right)^H \otimes \left(\check{q}^{\frac{\alpha-1}{4}}\right)^{2H} \otimes \cdots \otimes \left(\check{q}^{\frac{\alpha-1}{4}}\right)^{NH}. \quad (2.17)$$

Suppose that a knot  $\mathcal{K} \subset S^3$  is presented as a closure of a braid  $B_N$ . The colored Jones polynomial of  $\mathcal{K}$  can be calculated as a ‘quantum trace’ of  $\hat{B}_N$ . We choose the closure strands to go to the right of the braid. Denote by  $\left(\check{q}^{\frac{H}{2}}\right)^{\otimes N}$  an operator that acts as  $\check{q}^{\frac{H}{2}}$  on every module  $V_\alpha$  of  $V_\alpha^{\otimes N}$ . Then

$$J_\alpha(\mathcal{K}; \check{q}) = \check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} \text{Tr}_{V_\alpha^{\otimes N}} \left(\check{q}^{\frac{H}{2}}\right)^{\otimes N} \hat{B}_N, \quad (2.18)$$

here  $e(B_N)$  is the number of positive elementary braids minus the number of negative elementary braids in  $B_N$ . The prefactor  $\check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)}$  is the framing correction. It is due to the fact that the knot  $\mathcal{K}$  constructed by closing the braid  $B_N$ , has the blackboard framing  $e(B_N)$ .

## 2.3 Breaking a closure strand

We are going to use the  $SU(2)_q$ -invariance of the braid group representation in order to reduce the trace of eq. (2.18).

Consider a tangle constructed by closing all braid positions except the first one which we leave open. To this tangle we associate an operator  $\hat{B}_N^{(1)}$  acting on  $V_\alpha$  (the first element in

the product  $V_\alpha^{\otimes N}$ ):

$$\hat{B}_N^{(1)} = \text{Tr}_{V_\alpha^{\otimes(N-1)}} \left( I \otimes \left( \check{q}^{\frac{H}{2}} \right)^{\otimes(N-1)} \right) \hat{B}_N. \quad (2.19)$$

The operator  $I \otimes \left( \check{q}^{\frac{H}{2}} \right)^{\otimes(N-1)}$  in this formula acts as identity on the first  $V_\alpha$  and as  $\check{q}^{\frac{H}{2}}$  on all other  $V_\alpha$  of  $V_\alpha^{\otimes N}$ .  $\text{Tr}_{V_\alpha^{\otimes(N-1)}}$  is the trace taken over all  $V_\alpha$  of  $V_\alpha^{\otimes N}$  except the first one.

The  $SU_q(2)$  invariance of  $\hat{B}_N^{(1)}$  together with irreducibility of  $V_\alpha$  means that  $\hat{B}_N^{(1)}$  is proportional to the identity operator:

$$\hat{B}_N^{(1)} = CI, \quad C \in \mathbb{C}. \quad (2.20)$$

On the other hand, in view of eqs. (2.18) and (2.19), the Jones polynomial  $J_\alpha(\mathcal{K}; \check{q})$  is equal to the quantum trace of  $\hat{B}_N^{(1)}$  which corresponds to closing the remaining strand:

$$\begin{aligned} J_\alpha(\mathcal{K}; \check{q}) &= \check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} \text{Tr}_{V_\alpha} \check{q}^{\frac{H}{2}} \hat{B}_N^{(1)} = \check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} C \text{Tr}_{V_\alpha} \check{q}^{\frac{H}{2}} \\ &= \check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} C \frac{\check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}}}{\check{q}^{\frac{1}{2}} - \check{q}^{-\frac{1}{2}}}. \end{aligned} \quad (2.21)$$

Then, according to eqs. (1.1), (1.2),

$$V_\alpha(\mathcal{K}; \check{q}) = \check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} C. \quad (2.22)$$

To find the constant  $C$  we can choose any diagonal matrix element of  $\hat{B}_N^{(1)}$ . We will do it for  $f_0$ . Thus

$$V_\alpha(\mathcal{K}; \check{q}) = \check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} \text{Tr}_{f_0 \otimes V_\alpha^{\otimes(N-1)}} \left( I \otimes \left( \check{q}^{\frac{H}{2}} \right)^{\otimes(N-1)} \right) \hat{B}_N. \quad (2.23)$$

The symbol  $\text{Tr}_{f_0 \otimes V_\alpha^{\otimes(N-1)}}$  has the following meaning. The operator

$$\left( I \otimes \left( \check{q}^{\frac{H}{2}} \right)^{\otimes(N-1)} \right) \hat{B}_N \quad (2.24)$$

acts on the full space  $V_\alpha^{\otimes N}$ . We project this action onto the subspace  $f_0 \otimes V_\alpha^{\otimes(N-1)} \subset V_\alpha^{\otimes N}$  along the other subspaces  $f_m \otimes V_\alpha^{\otimes(N-1)}$ ,  $m \geq 1$ . Then we take the trace of this projection. In other words, we simply take the diagonal matrix element of (2.24) for  $f_0$  (with respect to our basis  $f_m$ ) in the first space  $V_\alpha$  and take traces over all other spaces  $V_\alpha$  of  $V_\alpha^{\otimes N}$ .

## 2.4 Extension and stratification of the trace

Our next step is to extend the  $SU_q(2)$  modules  $V_\alpha$  to the infinite-dimensional spaces  $V_{\alpha,\infty}$  by adding formally the basis vectors  $f_m$ ,  $m \geq \alpha$  to the already existing vectors  $f_m$ ,  $0 \leq m \leq \alpha - 1$ . The matrix elements of  $SU_q(2)$  generators  $X, Y, H$  are still defined by eqs. (2.1), (2.2) and (2.3), while the action of  $\check{R}$ -matrix is given by eq. (2.13).

The tensor product  $V_{\alpha,\infty}^{\otimes N}$  can be decomposed into a direct sum of linear spaces  $V_N^{(\eta)}$  corresponding to the eigenvalues  $\eta$  of the operator  $\frac{1}{2}[N(\alpha - 1) - H]$  acting on  $V_{\alpha,\infty}^{\otimes N}$ :

$$V_{\alpha,\infty}^{\otimes N} = \bigoplus_{\eta=0}^{\infty} V_N^{(\eta)}. \quad (2.25)$$

The subspace  $f_0 \otimes V_{\alpha,\infty}^{\otimes(N-1)}$  is also decomposed:

$$f_0 \otimes V_{\alpha,\infty}^{\otimes(N-1)} = \bigoplus_{\eta=0}^{\infty} f_0 \otimes V_{N-1}^{(\eta)}, \quad f_0 \otimes V_{N-1}^{(\eta)} \subset V_N^{(\eta)}. \quad (2.26)$$

The spaces  $V_N^{(\eta)}$ ,  $f_0 \otimes V_{N-1}^{(\eta)}$  are finite-dimensional.

Since  $H$  commutes with  $\check{R}$ -matrices (2.13) and with the operator  $I \otimes (\check{q}^{\frac{H}{2}})^{\otimes(N-1)}$ , we can split the trace (2.23) into the traces over the eigenspaces of  $H$ .

**Proposition 2.1** *For any  $M > 0$ ,*

$$V_\alpha(\mathcal{K}; \check{q}) = \check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} \check{q}^{\frac{1}{2}(\alpha-1)(N-1)} \sum_{0 \leq \eta \leq (N-1)\max\{\alpha-1, M\}} \check{q}^{-\eta} \text{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}} \hat{B}_N. \quad (2.27)$$

*Proof of Proposition 2.1.* First of all, since

$$f_0 \otimes V_\alpha^{\otimes(N-1)} \subset \bigoplus_{0 \leq \eta \leq (N-1)(\alpha-1)} f_0 \otimes V_{N-1}^{(\eta)} \quad (2.28)$$

and since  $f_0 \otimes V_{N-1}^{(\eta)}$  is an eigenspace of the operator  $I \otimes (\check{q}^{\frac{H}{2}})^{\otimes(N-1)}$  with the eigenvalue  $\check{q}^{\frac{1}{2}(\alpha-1)(N-1)} \check{q}^{-\eta}$ , we can transform eq. (2.23) into

$$V_\alpha(\mathcal{K}; \check{q}) = \check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} \check{q}^{\frac{1}{2}(\alpha-1)(N-1)} \sum_{0 \leq \eta \leq (N-1)\max\{\alpha-1, M\}} \check{q}^{-\eta} \text{Tr}_{f_0 \otimes V_\alpha^{\otimes(N-1)} \cap f_0 \otimes V_{N-1}^{(\eta)}} \hat{B}_N. \quad (2.29)$$

It remains to check that

$$\mathrm{Tr}_{f_0 \otimes V_\alpha^{\otimes(N-1)} \bigcap f_0 \otimes V_{N-1}^{(\eta)}} \hat{B}_N = \mathrm{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}} \hat{B}_N. \quad (2.30)$$

To show this consider a matrix element of the  $\check{R}$ -matrix between the vectors  $f_{m_1} \otimes f_{m_2}$  and  $f_{m_2+n} \otimes f_{m_1-n}$  such that  $0 \leq m_2 \leq \alpha - 1$  and  $m_2 + n \geq \alpha$ . In this case the product

$$\prod_{m_2+1 \leq l \leq m_2+n} (\check{q}^{-l} - \check{q}^{-\alpha}) \quad (2.31)$$

of eq. (2.13) contains a term corresponding to  $l = \alpha$ , so the matrix element is zero. Therefore if we follow the evolution of a vector *along the braid strand*, then we see that transitions from the elements  $f_m$ ,  $0 \leq m \leq \alpha - 1$  to  $f_m$ ,  $m \geq \alpha$  are forbidden at positive elementary braids. The same analysis of eq. (2.16) shows that these transitions are also forbidden at negative braids.

In the calculation of  $\mathrm{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}}$  the evolution of the vector along the braid strands and the closure strands starts at  $f_0$  at the beginning of the first strand. Following this evolution we will cover all the segments of the braid, because its closure is a knot. Therefore the elements  $f_m$ ,  $m \geq \alpha$  will never appear in the calculation of the *r.h.s.* of eq. (2.30).  $\square$

### 3 Expansion of $\check{R}$ -matrix

In order to use eq. (2.27) for calculation of the coefficients  $D_{m,n}$  we have to expand the matrix elements of  $\hat{B}_N$ , which appear in the *r.h.s.* of eq. (2.27), in powers of  $h$  at  $h = 0$ . These matrix elements come from the elements of  $\check{R}$ -matrices, so we have to study the expansion of eqs. (2.13) and (2.16). In accordance with Proposition 2.1 we fix a number  $M > 0$  and study the action of  $\check{R}$ -matrix only on those eigen-spaces  $V_2^{(\eta)} \subset V_{\alpha,\infty} \otimes V_{\alpha,\infty}$  for which  $0 \leq \eta \leq (N-1) \max\{\alpha-1, M\}$ . Therefore we assume that  $0 \leq m_1, m_2, n \leq (N-1) \max\{\alpha-1, M\}$  in eqs. (2.13) and (2.16). As a result, the expansion of matrix elements is achieved by substituting  $1+h$  instead of  $\check{q}$ :

$$\check{R}(f_{m_1} \otimes f_{m_2}) = \check{q}^{\frac{1}{4}(\alpha^2-1)} \check{q}^{-\frac{1}{2}(\alpha-1)} \sum_{n \geq 0} \prod_{m_2+1 \leq l \leq m_2+n} ((1+h)^{-l} - \check{q}^{-\alpha}) \quad (3.1)$$

$$\times \frac{\prod_{m_1-n+1 \leq l \leq m_1} ((1+h)^l - 1)}{\prod_{1 \leq l \leq n} ((1+h)^l - 1)} \check{q}^{-\alpha m_2} (1+h)^{m_2} f_{m_2+n} \otimes f_{m_1-n}$$

and expanding the resulting expression in powers of  $h$ .

We left  $\check{q}^\alpha$  intact in the *r.h.s.* of eq. (3.1). This gives us two options for expansion. An evaluation of the *r.h.s.* in eq. (2.27) requires us to expand in powers of  $h$  while keeping  $\alpha$  fixed. This means that  $1 - \check{q}^{-\alpha}$  is of order  $h$ . Then the coefficients in front of powers of  $h$  in the expansion of the matrix elements (3.1) can be expanded themselves in  $h$ . This is not dangerous for our purposes, because  $1 - \check{q}^{-\alpha}$  never appears in denominator, so any further expansion goes in positive powers of  $h$  and does not change our estimates of ‘smallness’. The second option is to keep  $1 - \check{q}^{-\alpha}$  constant and small as  $h \rightarrow 0$  and use eq. (3.1) to get at least a formal expansion in powers of  $h$  and  $1 - \check{q}^{-\alpha}$ . We will excercise this option in the end of Section 5 in order to prove the relation (1.15). Note that from the technical point of view both options lead to the same expansion formulas. Only the interpretation is different.

From now on we assume that  $\alpha$  is kept constant as  $h \rightarrow 0$ . Let us summarize the results of expanding the matrix elements (3.1) in powers of  $h$ . For every elementary positive braid in  $B_N$  we associate a parametrized  $\check{R}$ -matrix:

$$\begin{aligned} \check{\mathcal{R}}[a; \epsilon_1, \epsilon_2, \epsilon_{12}](f_{m_1} \otimes f_{m_2}) = & \sum_{n \geq 0} \frac{\prod_{m_1-n+1 \leq l \leq m_1} l}{n!} (e^{\epsilon_{12}} a)^n (e^{\epsilon_1})^{m_1} (e^{\epsilon_2} \check{q}^{-\alpha})^{m_2} \\ & \times f_{m_2+n} \otimes f_{m_1-n}. \end{aligned} \quad (3.2)$$

This  $\check{\mathcal{R}}$ -matrix depends on parameters  $a, \epsilon_1, \epsilon_2, \epsilon_{12}$  ‘attached’ to the elementary braid. For a negative elementary braid we associate the ‘inverse’ matrix

$$\begin{aligned} \check{\mathcal{R}}^{(\text{inv})}[a'; \epsilon_1, \epsilon_2, \epsilon_{12}](f_{m_1} \otimes f_{m_2}) = & \sum_{n \geq 0} \frac{\prod_{m_2-n+1 \leq l \leq m_2} l}{n!} (e^{\epsilon_{12}} a')^n (e^{\epsilon_1} \check{q}^\alpha)^{m_1} (e^{\epsilon_2})^{m_2} \\ & \times f_{m_2-n} \otimes f_{m_1+n}. \end{aligned} \quad (3.3)$$

Let us denote by  $\{a\}, \{a'\}, \{\epsilon\}$  the sets of parameters  $a, a'$  and  $\epsilon_1, \epsilon_2, \epsilon_{12}$  associated to all elementary braids in a particular expression of  $B_N$ . We construct a parametrized braid operator  $\hat{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}]$  acting on  $V_{\alpha, \infty}^{\otimes N}$ , from the matrices  $\check{\mathcal{R}}$  and  $\check{\mathcal{R}}^{(\text{inv})}$  in the same way as we constructed  $\hat{B}_N$  from  $\check{R}$  and  $\check{R}^{-1}$ . Since eqs. (3.2) and (3.3) do not constitute a

homomorphism of the braid group, the operator  $\hat{\mathcal{B}}_N$  depends on the choice of a presentation of  $B_N$  as a product of elementary braids.

**Proposition 3.1** *For a given braid  $B_N$  and its presentation as a product of elementary braids there exists a set of polynomials*

$$T_j(B_N | \{a\}, \{a'\}, \{\epsilon\}), \quad j \geq 1$$

such that

$$\deg T_j \leq 2j \quad (3.4)$$

and for a fixed  $\eta$  and for any positive  $M$

$$\begin{aligned} & \check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} \check{q}^{\frac{1}{2}(\alpha-1)(N-1)} \check{q}^{-\eta} \operatorname{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}} \hat{\mathcal{B}}_N \\ &= \check{q}^{\frac{1}{2}(\alpha-1)(N-1-e(B_N))} \left( 1 + \sum_{1 \leq j \leq M} h^j T_j(B_N | \{\partial_a\}, \{\partial_{a'}\}, \{\partial_\epsilon\}) \right) \left( 1 + \sum_{1 \leq j \leq M} h^j \frac{(-1)^j}{j!} \partial_\kappa^j \right) \\ & \quad \times \kappa^\eta \operatorname{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}} \hat{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}] \Big|_{\substack{\kappa=1 \\ \{a\}=1-\check{q}^{-\alpha} \\ \{a'\}=1-\check{q}^\alpha \\ \{\epsilon\}=0}} + \mathcal{O}(h^{M+1}). \end{aligned} \quad (3.5)$$

In our notations here  $\{\cdot\}$  means all elements of the set.

*Proof of Proposition 3.1.* Let us assume that the following lemma is true (see Appendix for the proof):

**Lemma 3.1** *There exist two sets of polynomials*

$$T_{j,k}^{(R,1)}(m, n), T_j^{(R,2)}(m, n) \in \mathbb{Q}[m, n], \quad \deg T_{j,k}^{(R,1)} \leq j+k, \quad \deg T_j^{(R,2)} \leq 2j \quad (3.6)$$

such that the expansion of eq. (3.1) in powers of  $h$  can be presented as

$$\begin{aligned} \check{R}(f_{m_1} \otimes f_{m_2}) &= \check{q}^{\frac{1}{4}(\alpha^2-1)} \check{q}^{-\frac{1}{2}(\alpha-1)} \sum_{n \geq 0} \left( 1 + \sum_{j \geq 1} h^j \frac{\prod_{0 \leq l \leq j-1} (n-l)}{(1-\check{q}^{-\alpha})^j} \sum_{k \geq 0} h^k T_{j,k}^{(R,1)}(m_2, m_2+n) \right) \\ & \quad \times \left( 1 + \sum_{j \geq 1} h^j T_j^{(R,2)}(m_1, n) \right) \left( 1 + \sum_{j \geq 1} h^j \frac{\prod_{0 \leq l \leq j-1} (m_2-l)}{j!} \right) \\ & \quad \times \frac{\prod_{m_1-n+1 \leq l \leq m_1} l}{n!} (1-\check{q}^{-\alpha})^n \check{q}^{-\alpha m_2} f_{m_2+n} \otimes f_{m_1-n}. \end{aligned} \quad (3.7)$$

A formula for  $\check{R}^{-1}$  can be obtained from eq. (3.7) by conjugation with  $P$  (*i.e.*, by the substitution (2.15)) and substitutions  $\check{q} \rightarrow \check{q}^{-1}$  and  $h \rightarrow \sum_{j \geq 1} (-1)^j h^j$ .

For the practical purpose of using our formulas for actual computation of the invariant polynomials  $V^{(n)}(\mathcal{K}; z)$  of specific knots we present the first polynomials  $T_{j,k}^{(R,1)}$  and  $T_j^{(R,2)}$ :

$$T_{1,0}^{(R,1)}(m_1, m_2) = -\frac{1}{2}(m_1 + m_2 + 1) \quad (3.8)$$

$$T_{1,1}^{(R,1)}(m_1, m_2) = \frac{1}{6}(m_1^2 + m_1 m_2 + m_2^2 + 3m_1 + 3m_2 + 2) \quad (3.9)$$

$$T_{2,0}^{(R,1)}(m_1, m_2) = \frac{1}{24}(3m_1^2 + 6m_1 m_2 + 3m_2^2 + 5m_2 + 7m_1 + 2) \quad (3.10)$$

$$T_1^{(R,2)}(m, n) = \frac{1}{2}(mn - n^2) \quad (3.11)$$

$$T_2^{(R,2)}(m, n) = \frac{1}{24}(3m^2 n^2 - 6mn^3 + 3n^4 + m^2 n - mn^2 - 5mn + 5n^2) \quad (3.12)$$

The variables  $m_1, m_2, n$  in the polynomials

$$T_{j,k}^{(R,1)}(m_2, m_2 + n), T_j^{(R,2)}(m_1, n), \prod_{0 \leq l \leq j-1} (m_2 - l) \quad (3.13)$$

can be switched to derivatives  $\partial_{\epsilon_1}, \partial_{\epsilon_2}, \partial_{\epsilon_{12}}$  with the help of relations

$$m_1 = \partial_{\epsilon_1} e^{\epsilon_1 m_1}|_{\epsilon_1=0}, \quad m_2 = \partial_{\epsilon_2} e^{\epsilon_2 m_2}|_{\epsilon_2=0}, \quad n = \partial_{\epsilon_{12}} e^{\epsilon_{12} n}|_{\epsilon_{12}=0}. \quad (3.14)$$

Another useful formula is

$$\frac{\prod_{0 \leq l \leq j-1} (n - l)}{(1 - \check{q}^{-\alpha})^j} (1 - \check{q}^{-\alpha})^n = \partial_a^j a^n \Big|_{a=1-\check{q}^{-\alpha}}. \quad (3.15)$$

A combination of eqs. (3.14) and (3.15) allows us to rewrite eq. (3.7).

**Corollary 3.1** *The expansion (3.7) can be expressed in terms of the parametrized matrix  $\check{\mathcal{R}}$ :*

$$\begin{aligned} \check{R} &= \check{q}^{\frac{1}{4}(\alpha^2-1)} \check{q}^{-\frac{1}{2}(\alpha-1)} \left( 1 + \sum_{j \geq 1} h^j \partial_a^j \sum_{k \geq 0} h^k T_{j,k}^{(R,1)}(\partial_{\epsilon_2}, \partial_{\epsilon_2} + \partial_{\epsilon_{12}}) \right) \\ &\times \left( 1 + \sum_{j \geq 1} h^j T_j^{(R,2)}(\partial_{\epsilon_1}, \partial_{\epsilon_{12}}) \right) \left( 1 + \sum_{j \geq 1} h^j \frac{\prod_{0 \leq l \leq j-1} (\partial_{\epsilon_2} - l)}{j!} \right) \\ &\times \check{\mathcal{R}}[a; \epsilon_1, \epsilon_2, \epsilon_{12}] \Big|_{\substack{a=1-\check{q}^{-\alpha} \\ \epsilon_1=\epsilon_2=\epsilon_{12}=0}}. \end{aligned} \quad (3.16)$$

The formula (3.5) follows easily from this corollary. The polynomials  $T_j$  are formed by the products of polynomials (3.13) coming from elementary positive braids and by the products of their counterparts coming from elementary negative braids. The factors  $\check{q}^{\frac{1}{4}(\alpha^2-1)}$  of individual  $\check{R}$ -matrices (3.16) are canceled by the factor  $\check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)}$  in the *l.h.s.* of eq. (3.5). The factors  $\check{q}^{-\frac{1}{2}(\alpha-1)}$  of individual  $\check{R}$ -matrices combine together with the factor  $\check{q}^{\frac{1}{2}(\alpha-1)(N-1)}$  from the *l.h.s.* of eq. (3.5) into the single factor  $\check{q}^{\frac{1}{2}(\alpha-1)(N-1-e(B_N))}$  in the *r.h.s.* of that equation. The factor  $\check{q}^{-\eta}$  in the *l.h.s.* of eq. (3.5) is represented by the factor

$$\left(1 + \sum_{1 \leq j \leq M} h^j \frac{(-1)^j}{j!} \partial_\kappa^j\right) \kappa^\eta \Big|_{\kappa=1} \quad (3.17)$$

in its *r.h.s.* . □

## 4 The trace of parametrized $\check{R}$ -matrix

The Proposition 3.1 presents the braid operator  $\hat{B}_N$  as a sum of derivatives of the parametrized braid operator  $\hat{\mathcal{B}}_N$ . Therefore calculating the sum

$$\sum_{\eta \geq 0} \kappa^\eta \text{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}} \hat{\mathcal{B}}_N \quad (4.1)$$

is a natural step towards evaluation of the *r.h.s.* of eq. (2.27).

The sum (4.1) can be expressed in terms of the action of  $\hat{\mathcal{B}}_N$  on the subspace  $V_N^{(1)} \subset V_{\alpha,\infty}^{\otimes N}$ . We denote this action as  $\tilde{\hat{\mathcal{B}}}_N$ . We express  $\tilde{\hat{\mathcal{B}}}_N$  as a product of matrices  $\check{\mathcal{R}}$  and  $\check{\mathcal{R}}^{(\text{inv})}$  acting on  $V_N^{(1)}$ .

The natural basis in the space  $V_N^{(1)}$  is formed by the vectors

$$e_j = f_0^{\otimes(j-1)} \otimes f_1 \otimes f_0^{\otimes(N-j)}, \quad 1 \leq j \leq N. \quad (4.2)$$

The matrix  $\check{\mathcal{R}}$  corresponding to the elementary positive braid  $\sigma_{j,j+1}$  transforms only the vectors  $e_j$  and  $e_{j+1}$ . This transformation is presented by a  $2 \times 2$  matrix

$$\tilde{\check{\mathcal{R}}}[a, \epsilon_1, \epsilon_2, \epsilon_{12}] = \begin{pmatrix} e^{\epsilon_1 + \epsilon_{12}} a & e^{\epsilon_2} \check{q}^{-\alpha} \\ e^{\epsilon_1} & 0 \end{pmatrix}. \quad (4.3)$$

The matrix  $\tilde{\mathcal{R}}^{(\text{inv})}$  corresponding to the negative braid  $\sigma_{j,j+1}^{-1}$  acts on the vectors  $e_j, e_{j+1}$  as

$$\tilde{\mathcal{R}}^{(\text{inv})}[a', \epsilon_1, \epsilon_2, \epsilon_{12}] = \begin{pmatrix} 0 & e^{\epsilon_2} \\ e^{\epsilon_1} \check{q}^\alpha & e^{\epsilon_2 + \epsilon_{12}} a' \end{pmatrix}. \quad (4.4)$$

Thus the matrix  $\tilde{\mathcal{B}}_N$  is a product of  $N \times N$  matrices containing  $2 \times 2$  blocks (4.3) and (4.4).

**Proposition 4.1** *For a braid  $B_N$  whose closure is a knot, there exist the constants*

$$\delta_\epsilon, \delta_a, \delta_{a'}, \delta > 0 \quad (4.5)$$

such that if

$$|\{\epsilon\}| < \delta_\epsilon, \quad |\{a\} - (1 - \check{q}^{-\alpha})| < \delta_a, \quad |\{a'\} - (1 - \check{q}^\alpha)| < \delta_{a'}, \quad |1 - \check{q}^\alpha| < \delta \quad (4.6)$$

(in our notations here  $\{\cdot\}$  means ‘for any element of the set’) and  $|\lambda| \leq 1$ , then

$$\sum_{\eta \geq 0} \lambda^\eta \text{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}} \hat{\mathcal{B}}_N = \frac{1}{\det_{f_0 \otimes V_{N-1}^{(1)}} (1 - \lambda \tilde{\mathcal{B}}_N)}, \quad (4.7)$$

here  $\det_{f_0 \otimes V_{N-1}^{(1)}}$  means that the matrix  $\tilde{\mathcal{B}}_N$  is projected onto a subspace  $f_0 \otimes V_{N-1}^{(1)} \subset V_N^{(1)}$  spanned by the vectors  $e_j$ ,  $2 \leq j \leq N$ , along the vector  $e_1$ , and the determinant is calculated in that subspace.

*Proof of Proposition 4.1.* The formula (4.7) follows from the fact that the operator  $\tilde{\mathcal{B}}_N$  acting on  $V_{\alpha, \infty}^{\otimes N}$  is an endomorphism of a symmetric tensor algebra generated by a linear transformation of the subspace  $V_N^{(1)}$ .

The choice of basis  $f_m$ ,  $m \geq 0$  for the spaces  $V_{\alpha, \infty}$  allows us to identify the tensor product  $V_{\alpha, \infty}^{\otimes N}$  with the algebra of polynomials  $\mathbb{C}[z_1, \dots, z_N]$ . The identification is achieved by mapping the basis vectors  $f_{m_1} \otimes \dots \otimes f_{m_N} \in V_{\alpha, \infty}^{\otimes N}$  to monomials  $z_1^{m_1} \cdots z_N^{m_N}$ . The subspaces  $V_N^{(\eta)}$  are mapped onto the polynomials of degree  $\eta$ .

Consider the action of parametrized  $\check{R}$ -matrix (3.2) on the algebra  $\mathbb{C}[z_1, z_2]$  which is isomorphic to  $V_{\alpha, \infty} \otimes V_{\alpha, \infty}$ .

**Lemma 4.1** *The action of  $\check{\mathcal{R}}$ -matrix (3.2) on  $\mathbb{C}[z_1, z_2]$  is an endomorphism which is generated by the linear transformation (4.3) of the variables  $z_1$  and  $z_2$ . The action of  $\check{\mathcal{R}}^{(\text{inv})}$  of eq. (3.2) is an endomorphism generated by the matrix (4.4) in the same way.*

*Proof of Lemma 4.1.* We will check this lemma only for  $\check{\mathcal{R}}$ . Consider a basis monomial  $z_1^{m_1} z_2^{m_2}$  corresponding to the basis vector  $f_{m_1} \otimes f_{m_2}$  of  $V_{\alpha, \infty} \otimes V_{\alpha, \infty}$ . The linear transformation (4.3) turns it into a polynomial

$$\begin{aligned} & (e^{\epsilon_1 + \epsilon_{12}} a z_1 + e^{\epsilon_1} z_2)^{m_1} (e^{\epsilon_2} \check{q}^{-\alpha} z_1)^{m_2} \\ &= \sum_{0 \leq n \leq m_1} \binom{m_1}{n} (e^{\epsilon_1 + \epsilon_{12}} a z_1)^n (e^{\epsilon_1} z_2)^{m_1 - n} (e^{\epsilon_2} \check{q}^{-\alpha} z_1)^{m_2} \\ &= \sum_{n \geq 0} \frac{\prod_{m_1 - n + 1 \leq l \leq m_1} l}{n!} (e^{\epsilon_{12}} a)^n (e^{\epsilon_1})^{m_1} (e^{\epsilon_2} \check{q}^{-\alpha})^{m_2} z_1^{m_2 + n} z_2^{m_1 - n}. \end{aligned} \quad (4.8)$$

The coefficients of the polynomial in the *r.h.s.* of this equation match the matrix elements of the  $\check{\mathcal{R}}$ -matrix (3.2).  $\square$

The parametrized matrix  $\hat{\mathcal{B}}_N$  is a product of matrices  $\check{\mathcal{R}}$  and  $\check{\mathcal{R}}^{(\text{inv})}$ , so it is also an endomorphism of  $\mathbb{C}[z_1, \dots, z_N]$  whose action is determined by the linear transformation  $\tilde{\hat{\mathcal{B}}}_N$  of the space of variables  $z_1, \dots, z_N$ . A projection of  $\hat{\mathcal{B}}_N$  onto a subspace  $f_0 \otimes V_{\alpha, \infty}^{\otimes(N-1)} \subset V_{\alpha, \infty}^{\otimes N}$  is an endomorphism of subalgebra  $\mathbb{C}[z_2, \dots, z_N] \subset \mathbb{C}[z_1, \dots, z_N]$ . This endomorphism is determined by the projection of  $\tilde{\hat{\mathcal{B}}}_N$  acting on the space  $V_N^{(1)}$  of  $N$  variables  $z_1, \dots, z_N$ , onto the subspace  $f_0 \otimes V_{N-1}^{(1)}$  of  $(N-1)$  variables  $z_2, \dots, z_N$ . Therefore the following well-known lemma proves eq. (4.7):

**Lemma 4.2** *Let  $\mathcal{O}$  be an endomorphism of the algebra  $\mathbb{C}[z_1, \dots, z_N]$  generated by its linear action  $\tilde{\mathcal{O}}$  on the space  $V_N^{(1)}$  of variables  $z_1, \dots, z_N$ . Suppose that for some  $\lambda \in \mathbb{C}$  the absolute values of eigen-values of the operator  $\lambda \tilde{\mathcal{O}}$  are smaller than 1. Denote by  $V_N^{(\eta)}$  the space of polynomials of degree  $\eta$ . Then*

$$\sum_{\eta \geq 0} \lambda^\eta \text{Tr}_{V_N^{(\eta)}} \mathcal{O} = \frac{1}{\det_{V_N^{(1)}} (1 - \lambda \tilde{\mathcal{O}})}. \quad (4.9)$$

An easy way to prove eq. (4.9) is to note that if  $\tilde{\mathcal{O}}$  is diagonalizable then eq. (4.9) becomes the formula for the sum of a convergent geometric series. Since diagonalizable matrices form a dense set in the space of all matrices, this proves the lemma.

It remains to check the applicability of the Lemma 4.2 to the formula (4.7). We have to verify that the eigenvalues of the operator  $\tilde{\mathcal{B}}_N$  are small enough. Note that the operator

$$\tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}] \Big|_{\substack{\{a\}=1 \\ \{a'\}=1 \\ \{\epsilon\}=0}} \quad (4.10)$$

acts on the space  $V_N^{(1)}$  by permuting the basis vectors  $e_1, \dots, e_N$  by the permutation corresponding to the braid  $B_N$ . Since the closure of  $B_N$  is a knot, this means that for any vector  $e_j$ ,  $2 \leq j \leq N$  some power of the operator (4.10) will map it to  $e_1$ . Therefore the projection of the operator (4.10) onto the space  $V_N^{(1)}$  is nilpotent. Thus all the eigenvalues of this projection are zero. Therefore the eigenvalues of the operator  $\tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}]$  are small as long as  $\{e\}$  are small,  $\{a\}$  are close to  $1 - \check{q}^{-\alpha}$ ,  $\{a'\}$  are close to  $1 - \check{q}^{\alpha}$  and  $1 - \check{q}^{\alpha}$  is kept small.  $\square$

## 5 The colored Jones polynomial and perturbed Bura representation

The Propositions 2.1, 3.1 and 4.1 bring us close to the following lemma:

**Lemma 5.1** *If a knot  $\mathcal{K} \subset S^3$  is presented as a closure of a braid  $B_N$ , then for any  $M > 0$*

$$\begin{aligned} \sum_{\substack{0 \leq n \leq M \\ 0 \leq m \leq \frac{n}{2}}} D_{m,n}(\mathcal{K}) \alpha^{2m} h^n &= \check{q}^{\frac{1}{2}(\alpha-1)(N-1-e(B_N))} \left( 1 + \sum_{1 \leq j \leq M} h^j T_j(B_N | \{\partial_a\}, \{\partial_{a'}\}, \{\partial_{\epsilon}\}) \right) \\ &\times \left( 1 + \sum_{1 \leq j \leq M} h^j \frac{(-1)^j}{j!} \partial_{\kappa}^j \right) \\ &\times \frac{1}{\det_{f_0 \otimes V_{N-1}^{(1)}} \left( 1 - \kappa \tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}] \right)} \Bigg|_{\substack{\kappa=1 \\ \{a\}=1-\check{q}^{-\alpha} \\ \{a'\}=1-\check{q}^{\alpha} \\ \{\epsilon\}=0}} + \mathcal{O}(h^{M+1}). \end{aligned} \quad (5.1)$$

*Proof of Lemma 5.1.* The only problem in combining eqs. (2.27), (3.5) and (4.7) is that in eq. (2.27) the sum over  $\eta$  goes up to  $(N-1) \max\{\alpha-1, M\}$  while in eq. (4.7) the sum

goes up to infinity. Therefore the lemma would follow from the following estimate: for any  $M > 0$

$$\left(1 + \sum_{1 \leq j \leq M} h^j T_j(B_N | \{\partial_a\}, \{\partial_{a'}\}, \{\partial_e\})\right) \left(1 + \sum_{1 \leq j \leq M} h^j \frac{(-1)^j}{j!} \partial_\kappa^j\right) \times \sum_{\eta > (N-1) \max\{\alpha-1, M\}} \kappa^\eta \text{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}} \tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}] \Bigg|_{\substack{\kappa=1 \\ \{a\}=1-\bar{q}^{-\alpha} \\ \{a'\}=1-\bar{q}^\alpha \\ \{\epsilon\}=0}} = \mathcal{O}(h^{M+1}). \quad (5.2)$$

The proof of eq. (5.2) requires the following lemma:

**Lemma 5.2** *For any  $M > 0$  the sum*

$$\sum_{\eta > (N-1)M} \kappa^\eta \text{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}} \tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}] \quad (5.3)$$

*is of combined order  $M + 1$  in variables  $\{a\}$  and  $\{a'\}$ .*

*Proof of Lemma 5.2.* In view of eq. (4.7),

$$\begin{aligned} \sum_{\eta > (N-1)M} \kappa^\eta \text{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}} \tilde{\mathcal{B}}_N &= \frac{1}{\det_{f_0 \otimes V_{N-1}^{(1)}} (1 - \kappa \tilde{\mathcal{B}}_N)} \\ &\quad - \sum_{0 \leq \eta \leq (N-1)M} \frac{1}{n!} \partial_\lambda^n \frac{1}{\det_{f_0 \otimes V_{N-1}^{(1)}} (1 - \kappa \lambda \tilde{\mathcal{B}}_N)} \Bigg|_{\lambda=0}. \end{aligned} \quad (5.4)$$

The *r.h.s.* of this equation is the error term in Taylor series expansion of the value of

$$\frac{1}{\det_{f_0 \otimes V_{N-1}^{(1)}} (1 - \kappa \lambda \tilde{\mathcal{B}}_N)} \quad (5.5)$$

at  $\lambda = 1$ . Therefore there exists  $\lambda_0 \in [0, 1]$  such that

$$\sum_{\eta > (N-1)M} \kappa^\eta \text{Tr}_{f_0 \otimes V_{N-1}^{(\eta)}} \tilde{\mathcal{B}}_N = \frac{1}{((N-1)M+1)!} \partial_\lambda^{(N-1)M+1} \frac{1}{\det_{f_0 \otimes V_{N-1}^{(1)}} (1 - \kappa \lambda \tilde{\mathcal{B}}_N)} \Bigg|_{\lambda=\lambda_0} \quad (5.6)$$

The determinant in the denominator of (5.5) is a polynomial in  $\kappa \lambda$  of degree  $\dim f_0 \otimes V_{N-1}^{(1)} = N-1$ :

$$\det_{f_0 \otimes V_{N-1}^{(1)}} (1 - \kappa \lambda \tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}]) = 1 + \sum_{1 \leq j \leq N-1} (\kappa \lambda)^j \tilde{Q}_j(\{a\}, \{a'\}, \{\epsilon\}), \quad (5.7)$$

here  $\tilde{Q}_j(\{a\}, \{a'\}, \{\epsilon\})$  are some functions. The matrix  $\tilde{\mathcal{B}}_N$  is formed by the products of matrices  $\tilde{\mathcal{R}}$  and  $\tilde{\mathcal{R}}^{(\text{inv})}$ . The matrix elements of (4.3) and (4.4) are products of variables  $a$ ,  $a'$ ,  $e^{\epsilon_1}$ ,  $e^{\epsilon_2}$ ,  $e^{\epsilon_{12}}$ ,  $\check{q}^\alpha$ ,  $\check{q}^{-\alpha}$ . As a result the matrix elements of  $\tilde{\mathcal{B}}_N$  are integer polynomials of these variables and

$$\tilde{Q}_j \in \mathbb{Z}[\{a\}, \{a'\}, \{e^\epsilon\}, \check{q}^\alpha, \check{q}^{-\alpha}]. \quad (5.8)$$

There is an anti-symmetric counterpart of the Lemma 4.2. Let  $\mathcal{O}$  be an operator acting in the  $N$ -dimensional space  $V_N$ . Then

$$\det(1 - \lambda \mathcal{O}) = \sum_{0 \leq j \leq N} (-1)^j \lambda^j \text{Tr}_{\bigwedge^j V_N} \mathcal{O}, \quad (5.9)$$

here  $\bigwedge^j V_N$  is the anti-symmetric part of  $V_N^{\otimes j}$  and the action of  $\mathcal{O}$  is canonically defined there.

It follows from eq. (5.9) that

$$\tilde{Q}_j(\{a\}, \{a'\}, \{\epsilon\}) = (-1)^j \text{Tr}_{\bigwedge^j(f_0 \otimes V_{N-1}^{(1)})} \tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}]. \quad (5.10)$$

We claim that the *l.h.s.* of this equation is of combined order 1 in  $\{a\}$  and  $\{a'\}$ . Indeed, eqs. (4.3) and (4.4) indicate that the matrix elements of the action of the operator

$$\tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}] \Big|_{\substack{\{a\}=0 \\ \{a'\}=0}} \quad (5.11)$$

on the basis vectors  $e_j$  of eq. (4.2) are proportional to the matrix elements of the permutation operator corresponding to the braid  $B_N$ . Therefore diagonal elements of the operator (5.11) with respect to the basis vectors  $e_{j_1} \wedge \cdots \wedge e_{j_m}$ ,  $m \leq N-1$  should be equal to zero (otherwise a set  $\{j_1, \dots, j_m\}$ ,  $m \leq N-1$  would be invariant under the permutation of  $B_N$  and the closure of  $B_N$  would be a link rather than a knot). Thus the value of the *r.h.s.* of eq. (5.10) at  $\{a\} = \{a'\} = 0$  is zero. Hence the polynomials  $\tilde{Q}_j$  are of combined order at least 1 in  $\{a\}$  and  $\{a'\}$ .

An application of  $(N-1)M+1$  derivatives over  $\lambda$  to the fraction

$$\frac{1}{\det_{f_0 \otimes V_{N-1}^{(1)}} \left( 1 - \kappa \lambda \tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}] \right)} = \frac{1}{1 + \sum_{1 \leq j \leq N-1} (\kappa \lambda)^j \tilde{Q}_j(\{a\}, \{a'\}, \{\epsilon\})} \quad (5.12)$$

brings the product of at least  $N$  polynomials  $\tilde{Q}_j$  to the numerator. Since each of them is of combined order 1 in  $\{a\}$  and  $\{a'\}$ , this proves the Lemma 5.2.  $\square$

We substitute  $\max\{\alpha - 1, M\}$  instead of  $M$  in the expression (5.3) in order to apply Lemma 5.2 to the proof of the estimate (5.2). The expression in the *l.h.s.* of eq. (5.2) contains derivatives over  $\{a\}$  and  $\{a'\}$  which reduce the powers of  $a$  and  $a'$  in (5.3). However, according to eq. (3.16), each derivative over  $a$  and  $a'$  is accompanied by a power of  $h$  in front of it. Ultimately we set  $\{a\} = 1 - \check{q}^{-\alpha}$ ,  $\{a'\} = 1 - \check{q}^{\alpha}$  which makes them small of order  $h$ . Since  $\max\{\alpha - 1, M\} \geq M$ , this proves the estimate (5.2) and completes the proof of eq. (5.1).  $\square$

Now we are ready to prove the following Proposition:

**Proposition 5.1** *For a knot  $\mathcal{K} \subset S^3$  there exist polynomials  $P_n(\check{q}^{\alpha}, \check{q}^{-\alpha}) \in \mathbb{Q}[\check{q}^{\alpha}, \check{q}^{-\alpha}]$  such that*

$$\sum_{\substack{0 \leq n \leq M \\ 0 \leq m \leq \frac{n}{2}}} D_{m,n}(\mathcal{K}) \alpha^{2m} h^n = \sum_{0 \leq n \leq M} h^n \frac{P_n((1+h)^{\alpha}, (1+h)^{-\alpha})}{\Delta_A^{2n+1}(\mathcal{K}; (1+h)^{\frac{\alpha}{2}} - (1+h)^{-\frac{\alpha}{2}})} + \mathcal{O}(h^{M+1}). \quad (5.13)$$

*Proof of Proposition 5.1.* The key observation is that the matrices

$$\tilde{\mathcal{R}}[a, \epsilon_1, \epsilon_2, \epsilon_{12}] \Big|_{\substack{a=1-\check{q}^{-\alpha} \\ \epsilon_1=\epsilon_2=\epsilon_{12}=0}} = \begin{pmatrix} 1 - \check{q}^{-\alpha} & \check{q}^{-\alpha} \\ 1 & 0 \end{pmatrix}, \quad (5.14)$$

$$\tilde{\mathcal{R}}^{(\text{inv})}[a', \epsilon_1, \epsilon_2, \epsilon_{12}] \Big|_{\substack{a'=1-\check{q}^{\alpha} \\ \epsilon_1=\epsilon_2=\epsilon_{12}=0}} = \begin{pmatrix} 0 & 1 \\ \check{q}^{\alpha} & 1 - \check{q}^{\alpha} \end{pmatrix} \quad (5.15)$$

coincide with the Burau matrix and its inverse respectively if we set

$$t = \check{q}^{-\alpha} \quad (5.16)$$

in the notations of [4]. Therefore the matrix

$$\tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}] \Big|_{\substack{\{a\}=1-\check{q}^{-\alpha} \\ \{a'\}=1-\check{q}^{\alpha} \\ \{\epsilon\}=0}} \quad (5.17)$$

reproduces the Burau representation of the braid group and

$$(\check{q}^{\alpha})^{\frac{1}{2}(e(B_N)-N+1)} \det_{f_0 \otimes V_{N-1}^{(1)}} \left( 1 - \kappa \tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}] \right) \Big|_{\substack{\kappa=1 \\ \{a\}=1-\check{q}^{-\alpha} \\ \{a'\}=1-\check{q}^{\alpha} \\ \{\epsilon\}=0}} = \Delta_A(\mathcal{K}; \check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}}) \quad (5.18)$$

(see eq. (5.22) of [4]).

The structure of the determinant in the *r.h.s.* of eq. (5.1) is described by eqs. (5.7), (5.8) (we have to set  $\lambda = 1$  there). The derivatives of the polynomial  $T_j(B_N | \{\partial_a\}, \{\partial_{a'}\}, \{\partial_\epsilon\})$ , as well as the derivatives  $\partial_\kappa^j$ , increase the power of the determinant in the denominator and bring the polynomials  $\tilde{Q}_j(\{a\}, \{a'\}, \{\epsilon\})$  and their derivatives to the numerator. The bound (3.4) limits the ‘extra’ powers of the determinant in the denominator by twice the power of  $h$ . Thus we recover the structure of the *r.h.s.* of eq. (5.13). The expansion of the prefactor

$$(\check{q}^\alpha)^{\frac{1}{2}(N-1-e(B_N))} (1+h)^{-\frac{1}{2}(N-1-e(B_N))} \quad (5.19)$$

in the *r.h.s.* of eq. (5.1) in powers of  $h$  does not change this structure because

$$N - 1 - e(B_N) \in 2\mathbb{Z} \quad (5.20)$$

if the closure of  $B_N$  is a knot.  $\square$

The Proposition 1.1 follows from the Proposition 5.1 and the next two lemmas which describe the properties of the polynomials  $P_n(\check{q}^\alpha, \check{q}^{-\alpha})$ .

**Lemma 5.3** *The polynomials  $P_n(\check{q}^\alpha, \check{q}^{-\alpha})$  can be reexpressed as*

$$P^{(n)}(\mathcal{K}; \check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}}) \in \mathbb{Q}[(\check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}})^2]. \quad (5.21)$$

**Lemma 5.4** *The polynomials  $P_n(\check{q}^\alpha, \check{q}^{-\alpha})$  have integer coefficients:*

$$P_n(\check{q}^\alpha, \check{q}^{-\alpha}) \in \mathbb{Z}[\check{q}^\alpha, \check{q}^{-\alpha}]. \quad (5.22)$$

*Proof of Lemma 5.3.* The *l.h.s.* of eq. (5.13) is an even function of  $\alpha$ . The Alexander-Conway polynomial of a knot  $\Delta_A(\mathcal{K}; \check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}})$  is also an even function of  $\alpha$ . Therefore the polynomials  $P_n(\check{q}^\alpha, \check{q}^{-\alpha})$  are even in  $\alpha$ . Since for any  $n \in \mathbb{Z}$ ,

$$\check{q}^{n\alpha} + \check{q}^{-n\alpha} = \mathbb{Z}[(\check{q}^{\frac{\alpha}{2}} - \check{q}^{-\frac{\alpha}{2}})^2], \quad (5.23)$$

this proves the Lemma 5.3.  $\square$

*Proof of Lemma 5.4.* Let us use eq. (3.1) in order to expand the matrix elements of  $\check{R}$  in powers of  $h$  as we keep  $1 - \check{q}^{-\alpha}$  small but fixed. We will apply the resulting formula towards the similar expansion of the trace

$$T_\eta(B_N) = \check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} \check{q}^{\frac{1}{2}(\alpha-1)(N-1)} \kappa^\eta \det_{f_0 \otimes V_{N-1}^{(\eta)}} \hat{B}_N \quad (5.24)$$

for the fixed value of  $\eta$ . In other words, we want to get an expansion

$$T_\eta(B_N) = \sum_{n \geq 0} h^n T_{\eta,n}(B_N | \check{q}^\alpha, \check{q}^{-\alpha}). \quad (5.25)$$

Let us first check that

$$T_{\eta,n}(B_N | \check{q}^\alpha, \check{q}^{-\alpha}) \in \mathbb{Z}[\check{q}^\alpha, \check{q}^{-\alpha}]. \quad (5.26)$$

Indeed, as we know, the factors  $\check{q}^{\frac{1}{4}(\alpha^2-1)} \check{q}^{-\frac{1}{2}(\alpha-1)}$  of the matrices (5.18) combine with the factor  $\check{q}^{-\frac{1}{4}(\alpha^2-1)e(B_N)} \check{q}^{\frac{1}{2}(\alpha-1)(N-1)}$  into a single factor  $\check{q}^{\frac{1}{2}(\alpha-1)(N-1-e(B_N))}$ . Then, according to (5.20),

$$\check{q}^{\frac{1}{2}(\alpha-1)(N-1-e(B_N))} = (\check{q}^\alpha)^{\frac{1}{2}(N-1-e(B_N))} (1+h)^{-\frac{1}{2}(N-1-e(B_N))} \in \mathbb{Z}[\check{q}^\alpha, \check{q}^{-\alpha}][[h]]. \quad (5.27)$$

As for the other parts of the matrix elements (3.1), the relation

$$\frac{\prod_{m_1-n+1 \leq l \leq m_1} ((1+h)^l - 1)}{\prod_{1 \leq l \leq n} ((1+h)^l - 1)} \in \mathbb{Z}[h] \quad (5.28)$$

shows that

$$\begin{aligned} \prod_{m_2+1 \leq l \leq m_2+n} ((1+h)^{-l} - \check{q}^{-\alpha}) & \frac{\prod_{m_1-n+1 \leq l \leq m_1} ((1+h)^l - 1)}{\prod_{1 \leq l \leq n} ((1+h)^l - 1)} \\ & \times \check{q}^{-\alpha m_2} (1+h)^{m_2} \in \mathbb{Z}[\check{q}^\alpha, \check{q}^{-\alpha}][[h]]. \end{aligned} \quad (5.29)$$

A similar relation holds for the elements of the inverse matrix  $\check{R}^{-1}$ . The remaining piece of the *r.h.s.* of eq. (5.24)

$$\kappa^\eta = (1+h)^{-\eta} \in \mathbb{Z}[[h]] \quad (5.30)$$

presents no problem. Thus we proved relation (5.26).

The Propositions 3.1, 4.1 allow us to construct a generating function of the traces (5.24) as a formal power series in  $h$ . More precisely, if we define a set of functions  $V_n(B_N|\check{q}^\alpha, \check{q}^{-\alpha}, \lambda)$  by the relation

$$\sum_{n \geq 0} h^n V_n(B_N|\check{q}^\alpha, \check{q}^{-\alpha}, \lambda) \quad (5.31)$$

$$\begin{aligned} &= (\check{q}^\alpha)^{\frac{1}{2}(N-1-e(B_N))} (1+h)^{-\frac{1}{2}(N-1-e(B_N))} \left( 1 + \sum_{1 \leq j \leq M} h^j T_j(B_N|\{\partial_a\}, \{\partial_{a'}\}, \{\partial_\epsilon\}) \right) \\ &\quad \times \left( 1 + \sum_{1 \leq j \leq M} h^j \frac{(-1)^j}{j!} \partial_\kappa^j \right) \\ &\quad \times \left. \frac{1}{\det_{f_0 \otimes V_{N-1}^{(1)}} \left( 1 - \kappa \lambda \tilde{\mathcal{B}}_N[\{a\}, \{a'\}, \{\epsilon\}] \right)} \right|_{\substack{\kappa=1 \\ \{a\}=1-\check{q}^{-\alpha} \\ \{a'\}=1-\check{q}^\alpha \\ \{\epsilon\}=0}}. \end{aligned} \quad (5.32)$$

then they generate the traces  $T_{\eta,n}$  by the formula

$$V_n(B_N|\check{q}^\alpha, \check{q}^{-\alpha}, \lambda) = \sum_{\eta \geq 0} \lambda^\eta T_{\eta,n}(B_N|\check{q}^\alpha, \check{q}^{-\alpha}), \quad (5.33)$$

The structure of the determinant in the denominator of the *r.h.s.* of eq. (5.31) is described again by eqs. (5.7), (5.8). Therefore, similar to the proof of the Proposition 5.1, we observe that the derivatives in the *l.h.s.* of eq. (5.33) add to the power of denominator and bring the polynomials  $\tilde{Q}_j$  and their derivatives to the numerator. Thus we conclude that there exist the polynomials

$$Q_{n,k}(\check{q}^\alpha, \check{q}^{-\alpha}) \in \mathbb{Q}[\check{q}^\alpha, \check{q}^{-\alpha}] \quad (5.34)$$

such that

$$V_n(B_N|\check{q}^\alpha, \check{q}^{-\alpha}, \lambda) = \frac{\sum_{0 \leq k \leq 2n(N-1)} \lambda^k Q_{n,k}(\check{q}^\alpha, \check{q}^{-\alpha})}{\left( 1 + \sum_{1 \leq j \leq N-1} \lambda^j Q_j(\check{q}^\alpha, \check{q}^{-\alpha}) \right)^{2n+1}}, \quad (5.35)$$

here

$$Q_j(\check{q}^\alpha, \check{q}^{-\alpha}) = \tilde{Q}_j(\{a\}, \{a'\}, \{\epsilon\}) \Big|_{\substack{\{a\}=1-\check{q}^{-\alpha} \\ \{a'\}=1-\check{q}^\alpha \\ \{\epsilon\}=0}} \in \mathbb{Z}[\check{q}^\alpha, \check{q}^{-\alpha}]. \quad (5.36)$$

The bound  $0 \leq k \leq 2n(N - 1)$  on the power of  $\lambda$  in the numerator comes from the fact that the maximum power of  $\lambda$  in denominator is  $N - 1$ , each derivative brings one term  $\lambda^j \tilde{Q}_j$  to the numerator and the maximum number of derivatives at  $h^n$  in the *r.h.s.* of eq. (5.31) is  $2n$ .

The generating function  $V_n(B_N | \check{q}^\alpha, \check{q}^{-\alpha}, \lambda)$  of eq. (5.33) satisfies the property that

$$\frac{1}{\eta!} \partial_\lambda^\eta V_n(B_N | \check{q}^\alpha, \check{q}^{-\alpha}, \lambda) \Big|_{\lambda=0} = T_{\eta, n}(B_N | \check{q}^\alpha, \check{q}^{-\alpha}). \quad (5.37)$$

This equation together with the relation (5.26) and eq. (5.35) implies that for all  $\eta \geq 0$

$$\frac{1}{\eta!} \partial_\lambda^\eta \left. \frac{\sum_{0 \leq k \leq 2n(N-1)} \lambda^k Q_{n,k}(\check{q}^\alpha, \check{q}^{-\alpha})}{\left(1 + \sum_{1 \leq j \leq N-1} \lambda^j Q_j(\check{q}^\alpha, \check{q}^{-\alpha})\right)^{2n+1}} \right|_{\lambda=0} \in \mathbb{Z}[\check{q}^\alpha, \check{q}^{-\alpha}]. \quad (5.38)$$

A combination of relations (5.36) and (5.38) means that

$$Q_{n,k}(\check{q}^\alpha, \check{q}^{-\alpha}) \in \mathbb{Z}[\check{q}^\alpha, \check{q}^{-\alpha}]. \quad (5.39)$$

Indeed, for a given  $n$  let  $k_0$  be the smallest value of  $k$  for which (5.39) is not true. Then the relation (5.38) is not satisfied for  $\eta = k_0$ .

Comparing eqs. (5.1), (5.13), (5.31) and (5.35), we see that

$$P_n(\check{q}^\alpha, \check{q}^{-\alpha}) = \sum_{0 \leq k \leq 2n(N-1)} Q_{n,k}(\check{q}^\alpha, \check{q}^{-\alpha}). \quad (5.40)$$

This together with the relation (5.39) proves the Lemma 5.4.  $\square$

This completes the proof of Proposition 1.1.

## 6 Discussion

The method of calculation of the coefficients  $V^{(n)}(\mathcal{K}; z)$  in the expansion (1.12) of the colored Jones polynomial, which we suggested in this paper, requires a presentation of the knot  $\mathcal{K}$  as a closure of a braid. However, there are  $R$ -matrix formulas for the colored Jones polynomial of a general blackboard picture of a knot. A direct calculation of the coefficients in this case

requires a certain generalization. The present method is based on a formula for the trace of an endomorphism of a symmetric algebra. In case of a blackboard picture, the number of strands is changing as one moves upwards. Our calculation can be adapted for this case if we use the so-called ‘holomorphic representation’ of the homomorphisms of symmetric algebras. This representation presents the homomorphisms as integral operators with gaussian kernels acting on the algebra of polynomials.

The Melvin-Morton expansion of the colored Jones polynomial comes in the holomorphic representation approach from the stationary phase approximation applied to a certain finite-dimensional integral. We find this setting a bit more universal than taking derivatives of an inverse determinant, as we have done here.

A slight modification of the procedure which led to the *r.h.s.* of eq. (5.1), can be applied to links. From the purely technical point of view, this is an almost straightforward exercise. This calculation produces invariants of links. We expect these invariants to reproduce the Melvin-Morton expansion of the colored Jones polynomial of a link only after a special ‘step-by-step’ procedure.

The relation between the new invariants of links and the colored Jones polynomial can be explained (at the level of a conjecture) with the help of path integral arguments. Let

$$\check{q} = e^{\frac{2\pi i}{K}}, \quad K \in \mathbb{Z}. \quad (6.1)$$

Consider the colored polynomial of a link in the limit of  $K \rightarrow \infty$ , while we keep the ratios

$$a_j = \frac{\alpha_j}{K} \quad (6.2)$$

( $\alpha_j$  being the colors of the link components) constant. According to [3], the Jones polynomial in this limit can be presented as a path integral of the exponential of the Chern-Simons action. The integration should go over all  $SU(2)$  connections  $A_\mu$  in the link complement which satisfy the boundary condition: up to a conjugation, in the fundamental representation of  $SU(2)$

$$\text{Pexp} \left( \oint_{C_j} A_\mu dx^\mu \right) = \exp \left[ i\pi a_j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right], \quad (6.3)$$

here  $\text{Pexp} \left( \oint_{\mathcal{C}_j} A_\mu dx^\mu \right)$  is a physical notation for the holonomies of the connection  $A_\mu$  along the meridians  $\mathcal{C}_j$  of the link components.

In the limit of  $K \rightarrow \infty$  the path integral can be calculated in the stationary phase approximation. The stationary points are flat connections in the link complement which satisfy the boundary conditions (6.3). In case of a knot, if  $a = \frac{\alpha}{K}$  is small enough, then there is only one flat connection which satisfies (6.3). Its contribution generates the expansion (1.12). This connection is reducible, all its holonomies lie in the same subgroup  $U(1) \subset SU(2)$ . This reducibility corresponds to the fact that the expansion (1.12) came in our calculations from the contribution of the ‘edge’ of the space  $V_\alpha^{\otimes N}$ . Within the path integral interpretation, the ‘top’ of  $V_\alpha^{\otimes N}$  corresponds to the situation when all the holonomies are aligned along the same subgroup  $U(1) \subset SU(2)$ .

In case of a link, generally there are flat irreducible  $SU(2)$  connections in the link complement satisfying the condition (6.3) even if the phases  $a_j$  are arbitrarily small. Therefore we do not expect that the application of eq. (5.1) to a braid whose closure is a link, would yield the actual expansion of the colored Jones polynomial of that link. We conjecture that the *r.h.s.* of eq. (5.1) gives an expansion of the contribution of a reducible connection to the Jones polynomial. This contribution was discussed in [11] in the context of the Reshetikhin formula for the Jones polynomial. It seems that this contribution is an invariant of the link in itself. We also think that it determines the perturbative invariants as well as  $p$ -adic properties of the Witten-Reshetikhin-Turaev invariant of rational homology spheres constructed by a surgery on the link.

We will address the issues of the blackboard calculations, holomorphic representation and new invariants of links in the future publication.

## Acknowledgements

I am thankful to D. Bar-Natan, J. Birman, P. Deligne, V. Drinfeld, D. Freed, S. Garoufalidis, L. Kauffman, M. Kontsevich, P. Melvin, R. Nepomechie, D. Thurston, V. Turaev,

A. Vaintrob, A. Varchenko, Z. Wang, E.Witten and C. Zachos for many useful discussions.

This work was supported by the National Science Foundation under Grants No. PHY-92 09978 and DMS 9304580. Part of this work was done during my visit to Argonne National Laboratory. I want to thank Professor C. Zachos for his hospitality.

## Appendix

*Proof of Lemma 3.1.* This lemma follows from the next two lemmas.

**Lemma A.1** *There exist the polynomials  $T_j^{(R,2)}(m, n)$ ,  $\deg T_j^{(R,2)} \leq 2j$  such that*

$$\frac{\prod_{m-n+1 \leq l \leq m} ((1+h)^l - 1)}{\prod_{1 \leq l \leq n} ((1+h)^l - 1)} = \frac{\prod_{m-n+1 \leq l \leq m} l}{n!} \left( 1 + \sum_{j \geq 1} h^j T_j^{(R,2)}(m, n) \right). \quad (\text{A.1})$$

*Proof of Lemma A.1.* It is enough to prove that there exist the polynomials  $\tilde{T}_j^{(R,2)}(n)$ ,  $\deg \tilde{T}_j^{(R,2)} \leq 2j$  such that

$$\prod_{1 \leq l \leq n} \frac{(1+h)^l - 1}{h} = n! \left( 1 + \sum_{j \geq 1} h^j \tilde{T}_j^{(R,2)}(n) \right). \quad (\text{A.2})$$

Lemma A.1 would follow because the *l.h.s.* of eq. (A.1) can be presented as a combination of three products of the type (A.2).

Consider the logarithm of the *l.h.s.* of eq. (A.2):

$$\begin{aligned} \log \left( \prod_{1 \leq l \leq n} \frac{(1+h)^l - 1}{h} \right) &= \log(n!) + \sum_{1 \leq l \leq n} \log \left( 1 + \sum_{k \geq 1} h^k \frac{\prod_{0 \leq m \leq k} (l-m)}{(k+1)!} \right) \\ &= \log(n!) - \sum_{1 \leq l \leq n} \sum_{k' \geq 1} \frac{(-1)^{k'}}{k'} \left( \sum_{k \geq 1} h^k \frac{\prod_{0 \leq m \leq k} (l-m)}{(k+1)!} \right)^{k'}. \end{aligned} \quad (\text{A.3})$$

The coefficient at a given power  $h^j$  in the *r.h.s.* of eq. (A.3) is a polynomial in  $l$  of degree  $j$ . The sum  $\sum_{1 \leq l \leq n}$  turns it into a polynomial in  $n$  of degree  $j+1$ . Since  $j \geq 1$ , this degree is not greater than  $2j$ . Taking the exponential of eq. (A.3) and expanding it in powers of  $h$  gives us the expansion (A.1).  $\square$

**Lemma A.2** *There exist the polynomials  $T_{j,k}^{(R,1)}(m_1, m_2)$ ,  $\deg T_{j,k}^{(R,1)} \leq j + k$ , such that*

$$\prod_{m+1 \leq l \leq m+n} ((1+h)^{-l} - \check{q}^{-\alpha}) = \left( 1 + \sum_{j \geq 1} h^j \frac{\prod_{0 \leq l \leq j-1} (n-l)}{(1-\check{q}^{-\alpha})^j} \sum_{k \geq 0} h^k T_{j,k}^{(R,1)}(m, m+n) \right) \times (1-\check{q}^{-\alpha})^n. \quad (\text{A.4})$$

*Proof of Lemma A.2.* We expand the *l.h.s.* of eq. (A.4) in powers of  $(1+h)^{-l} - 1$ :

$$\begin{aligned} \prod_{m+1 \leq l \leq m+n} ((1+h)^{-l} - \check{q}^{-\alpha}) &= \prod_{m+1 \leq l \leq m+n} [((1+h)^{-l} - 1) - (1-\check{q}^{-\alpha})] \\ &= (1-\check{q}^{-\alpha})^n \sum_{j \geq 0} \frac{h^j}{(1-\check{q}^{-\alpha})^j} S_j(m, m+n; h). \end{aligned} \quad (\text{A.5})$$

$$S_j(m_1, m_2; h) = \sum_{m_1+1 \leq l_1 < \dots < l_j \leq m_2} \prod_{1 \leq i \leq j} \frac{(1+h)^{-l_i} - 1}{h}. \quad (\text{A.6})$$

Now it remains to prove the following:

**Lemma A.3** *There exist the polynomials  $T_{j,k}^{(R,1)}(m_1, m_2)$ ,  $\deg T_{j,k}^{(R,1)} \leq j + k$ , such that*

$$S_j(m_1, m_2; h) = \left( \prod_{0 \leq i \leq j-1} (m_2 - m_1 - i) \right) \sum_{k \geq 0} h^k T_{j,k}^{(R,1)}(m_1, m_2). \quad (\text{A.7})$$

*Proof of Lemma A.3.* We prove this lemma by induction. When  $j = 0$ , the lemma is obvious:

$$S_0(m_1, m_2; h) = 1, \quad T_{0,k}(m_1, m_2) = \delta_{0,k}. \quad (\text{A.8})$$

Suppose that the lemma is true for a particular value of  $j$ . The sum  $S_{j+1}(m_1, m_2; h)$  can be expressed in terms of  $S_j$ :

$$S_{j+1}(m_1, m_2; h) = \sum_{m_1+1 \leq l \leq m_2} S_j(m_1, l-1; h) \frac{(1+h)^{-l} - 1}{h}. \quad (\text{A.9})$$

Expanding the last factor

$$\frac{(1+h)^{-l} - 1}{h} = \sum_{k \geq 0} h^k \frac{\prod_{0 \leq i \leq k} (l+i)}{(k+1)!} \quad (\text{A.10})$$

and substituting the formula (A.7) for  $S_j(m_1, l - 1; h)$  we find that

$$S_{j+1}(m_1, m_2; h) = \sum_{k \geq 0} h^k \sum_{0 \leq k' \leq k} T_{j+1, k, k'}^{(1)}(m_1, m_2), \quad (\text{A.11})$$

$$T_{j+1, k, k'}^{(1)}(m_1, m_2) = \sum_{m_1+1 \leq l \leq m_2} \prod_{1 \leq i \leq j} (l - m_1 - i) \frac{\prod_{0 \leq i \leq k-k'}(l+i)}{(k-k'+1)!} T_{j, k'}^{(R, 1)}(m_1, l-1). \quad (\text{A.12})$$

The *r.h.s.* of eq. (A.12) contains finite sums of polynomials of  $m_1, l$ , so it is clear that  $T_{j+1, k, k'}^{(1)}$  is a polynomial in  $m_1, m_2$  such that

$$\deg T_{j+1, k, k'}^{(1)} \leq 2(j+1) + k. \quad (\text{A.13})$$

The factor  $\prod_{1 \leq i \leq j} (l - m_1 - i)$  in eq. (A.12) guarantees that

$$T_{j+1, k, k'}^{(1)}(m_1, m_2) = 0 \quad \text{if } m_2 - m_1 \in \mathbb{Z}, 1 \leq m_2 - m_1 \leq j. \quad (\text{A.14})$$

Denote by  $B_n(m_1, m_2)$  a polynomial

$$B_n(m_1, m_2) = \sum_{m_1 \leq m \leq m_2} m^n. \quad (\text{A.15})$$

Since  $B_n(m_1, m_2) = 0$  for  $m_1 = m_2$ , we conclude from eq. (A.12) that

$$T_{j+1, k, k'}^{(1)}(m_1, m_2) = 0 \quad \text{if } m_2 - m_1 = 0. \quad (\text{A.16})$$

Equations (A.14) and (A.16) imply together that there exist the polynomials  $\tilde{T}_{j+1, k, k'}^{(1)}(m_1, m_2)$  such that

$$T_{j+1, k, k'}^{(1)}(m_1, m_2) = \left( \prod_{0 \leq i \leq j} (m_2 - m_1 - l) \right) \tilde{T}_{j+1, k, k'}^{(1)}(m_1, m_2). \quad (\text{A.17})$$

Then the polynomials

$$T_{j+1, k}^{(R, 1)}(m_1, m_2) = \sum_{0 \leq k' \leq k} \tilde{T}_{j+1, k, k'}^{(1)}(m_1, m_2) \quad (\text{A.18})$$

satisfy eq. (A.7) at  $j+1$ . They also have the right degree in view of eq. (A.13). This proves Lemma A.3.  $\square$

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